Positive Tree-like Mapping Classes

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Introduction

In this paper we will study special mapping classes of orientable surfaces with one boundary component. A mapping class is an element of the mapping class group. The mapping class group for an orientable surface S with boundary is defined as

$$MCG(S) = \mathrm{Diff}^+(S, \partial S)/\mathrm{Diff}_0^+(S, \partial S).$$

So a mapping class is an isotopy class of diffeomorphisms fixing the boundary pointwise. Tree-like mapping classes are represented by diffeomorphisms, that are a product of Dehn twists along a system of essential simple closed curves on the surface. Curves of the system intersect at most once with another curve and the complement of the system of curves is a cylindrical neighborhood of the boundary. We can build a graph by representing each twist curve by a vertex and connecting two vertices by an edge if the two corresponding curves intersect. The mapping class is called *tree-like*, if this graph is a tree. When we keep the information of cyclic ordering of the curves on the surface, we get a planar tree, and call it the *geometrical Dynkin diagram*.

A tree-like mapping class is called *positive*, if all Dehn twists which are performed are right or positive Dehn twists.

We will establish an algorithm to distinguish positive tree-like mapping classes up to conjugacy. The conjugacy problem for surface mapping classes has already been solved by Thurston, but in concrete examples it can be very hard to determine whether two mapping classes are conjugate or not.

In the following we will speak of surface diffeomorphisms meaning surface diffeomorphisms up to isotopy and mapping classes, respectively. So we will define a diffeomorphism and regard it as a representative for a mapping class.

Positive tree-like diffeomorphisms arise as monodromies of a special class of fibred knots, the slalom knots. These slalom knots can be constructed out of a rooted planar tree, which is related to the geometrical Dynkin diagram.

Up to the exceptions E_6, E_8 and the series A_{2n} the monodromies of slalom knots are pseudo-Anosov. By a theorem of Thurston, there exist two transverse measured foliations that are invariant under this diffeomorphism. In chapter 3 we will give an explicit description of the measured foliations for slalom knots with pseudo-Anosov monodromies. The rooted planar tree will play again a crucial role. Measured foliations have been studied under different aspects. Casson and Bleiler considered in [**BC**] geodesic lamination and Strebel studied in [**S**] quadratic differentials. For further studies see [**FPL**].

The two measured foliations of a monodromy are an invariant of the diffeomorphism up to conjugacy. But since measured foliations contain a lot of information that is not easy to take care of, it is very hard to use it as a tool to distinguish concrete diffeomorphisms.

Particularly, it is very hard to distinguish slalom monodromies that arise from the same abstract rooted tree, but from different embeddings into the plane since the corresponding slalom knots are mutant. The notion of mutation was introduced by Conway in [**Co**]. Mutant knots are hard to distinguish. For small examples the quantum invariant can be calculated and separates. Knotscape too, helps us to separate small examples. Sometimes there is also a symmetry argument that can be applied. But for the whole class of slalom knots it was not known if all knots coming from different rooted planar trees were different.

In chapter 5 we give a solution to this problem. We introduce a method to reconstruct the rooted planar tree out of the diffeomorphism by a geometrical algorithm for all diffeomorphisms that arise from rooted planar trees with at least three crown vertices. So the rooted planar tree is an invariant for the slalom knot, and hence all slalom knots are different. Slalom knots with one or two crown vertices arise from trees with only one planar embedding. The theory of the Montesinos links can be applied to them, and separates them (see [**Tu**]). Therefore we obtain the result, that all slalom knots coming from non-congruent planar trees are different.

To get this algorithm, we need an important property of the slalom monodromy. All slalom monodromies are strongly inversive. This means, that there exists an involution, that conjugates the monodromy to its inverse. This property is inherited from slalom knot. Slalom knots are strongly invertible, so there exists an involution of S^3 sending the oriented knot to itself, fixing two points on the knot, and reversing the orientation of the knot. If the knot is fibred, this involution can be chosen to respect the fibers, and therefore the monodromy becomes strongly inversive [**To**].

In chapter 4 we analyse these involutions. We will see, that up to conjugacy of the pair (monodromy, involution), there are at most two such involutions. Each of these involutions fixes an arc on the surface pointwise. We will study these fixed arcs and their images under the monodromy, and we will see, that the number of intersections of the fixed arc and its image under the monodromy differ for the two arcs coming from the two involutions. So the two involutions can be distinguished using their fixed arcs. Furthermore, one of this fixed arcs will play a crucial role in the reconstruction of the rooted planar tree.

Contents

CHAPTER 1

Preparations

1. Slalom knots, Fiber Surfaces, Monodromy

Slalom knots belong to the class of arborescent knots. They can be obtained by plumbing positive Hopf bands, where the information of plumbing is contained in a planar tree.

Here we chose another method to obtain a slalom knot. We start with a rooted planar tree B. A rooted tree is a tree, with a marked vertex of valence one, called the *root*. The tree is embedded into the unit disk D, such that $B \cap D$ contains only the root of the tree.

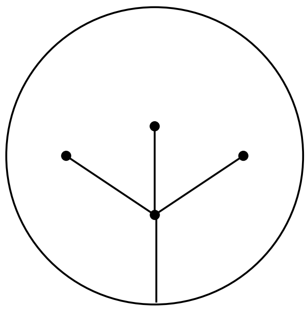

Figure 1: A rooted tree embedded in the unit disk

With the tree embedded in the unit disk, we can draw an immersed generic copy of the unit interval, called *divide curve*. Locally on an edge, we have the following pictures:

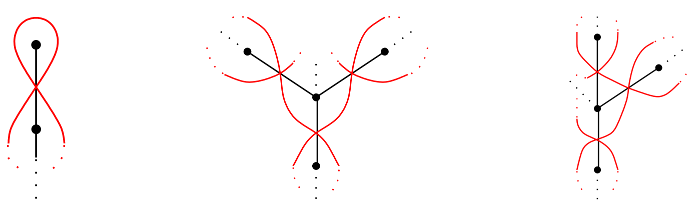

Figure 2: Local pictures of the divide curve

The planar tree gives now the information how the local pieces are put together. The curve we get is called divide curve P_B. For more detail see [**AC2**].

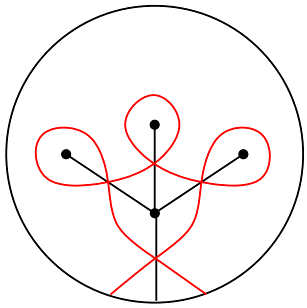

Figure 3: The divide curve P_B (red)

With a generic relative immersed copy of the unit interval P in the unit disk there can be constructed a knot $K \subset S^3$ in the following way:

$$K := \{(x,u) \,\epsilon\, \mathbb{R}^4 |\; x \,\epsilon\, P,\; u \,\epsilon\, T_xP,\; \|x\|^2 + |u\|^2 = 1\} \;\subset\; S^3.$$

Divide knots have many special properties. The property of interest for us is that they are all strongly invertible (see [**K**]). The inversion is given by $(x,u) \mapsto (x,-u)$.
When the divide curve P_B comes from a rooted planar tree B, then the knot we obtain is called a *slalom knot*. Every knot constructed out of a connected divide curve is fibred and hence every slalom knot is fibred. For the following we will only need the fact, that a slalom knot is fibred. We will not work with the knot itself, only with the fiber surface and the monodromy. The knot diagram can be algorithmically constructed out of the rooted tree, see [**H**]. This algorithm generates the knot diagrams, but they are not minimal. Ishikawa gave in [**I**] a suggestion for a minimal diagram, and Baader proved its minimality in [**Ba**].

The *geometrical Dynkin diagram* of a slalom knot is obtained by the rooted planar tree. Each edge except the edge containing the root is subdivided by a new vertex and at each vertex we have the information of the cyclic ordering of the edges around that vertex, i.e. the geometrical Dynkin diagram contains the information of the planar embedding of the abstract tree.

With the geometrical Dynkin diagram we can construct the fiber surface together with a system of simple closed curves on it. Each vertex gives a cycle and the edges between the vertices contain the information if two cycles intersect. The neighborhood of these $2g$ cycles is our fiber surface. It is a surface of genus g with one boundary component. The $2g$ cycles can be split into two groups, the A-curves or A-cycles and the B-curves or the B-cycles. The A-cycles are those who correspond to the edges in the original rooted tree, the B-curves correspond to the vertices except the root in the planar tree. So we have g A-curves and g B-curves.

The monodromy can be written as a product of right Dehn twists along the A- and B-curves. A right Dehn twist along a simple closed curve γ is a homeomorphism defined as follows. Take a regular neighborhood N of γ. N is an annulus and homeomorphic to $S^1 \times [0,1]$ oriented by the induced orientation. Give $S^1 \times [0,1]$ the coordinates (θ, t). Then the right Dehn twist D is the identity outside N and on N it is the map $D(\theta, t) := (\theta - e^{2\pi i t}, t)$.

Figure 4: On the left a cylinder before and on the right a cylinder after a right Dehn twist

We call T_A the diffeomorphism which is obtained, when we perform on each A-curve a right Dehn twist, and analogous for T_B. Since all A-curves are pairwise disjoint (and analogous for the B-curves), the order in which the twists are performed can be chosen arbitrarily. We define the monodromy diffeomorphism T as:

$$T := T_A \circ T_B$$

So T is a product of $2g$ right Dehn twists. We have chosen an order in which the the twist are performed. This choice is arbitrary, since all diffeomorphisms which result in performing these Dehn twist in any order are all conjugate (see [**Bo**]).

1.1. Nomenclature on a Rooted Tree. In the following, we will need appropriate names for the different edges and vertices of a rooted tree.

Definition 1.1. The below nomenclature will be used:

- An edge which originates in the root is called *trunk edge* or only *trunk*, and the other adjacent vertex is called the *trunk vertex*.
- A vertex which has only one adjacent edge (except the root) is called a *crown vertex*
- A vertex which has more than one adjacent edge, except the trunk vertex, is called an *interior vertex*
- At a vertex v, the adjacent edge is called *crown-sided*, if it is not contained in the path from v to the root.
- At an edge e, the adjacent vertex v is called *root-sided*, if e is crown-sided for v.

1.2. The tree vector β. We will use a vector to represent a rooted planar tree. First we have to enumerate the vertices and the edges in the planar tree. The root gets the number 0, the trunk vertex is numerated by 1. Next, we enumerate all vertices in the tree, that have distance 2 to the root. We numerate them such that the numbers increase from left to right. In a next step all vertices are numerated, that have distance 3 to the root. We enumerate the tree level by level. To each vertex, except the root, there is assigned exactly one edge, namely the one, that is the first edge from that vertex on a path to the root. The vertex and its edge are labelled by the same number.
The tree vector β is defined as follows. The i-th entry of the vector is the number of the root-sided vertex of the edge with number i. The tree vector for the tree in figure 1 is [0111].

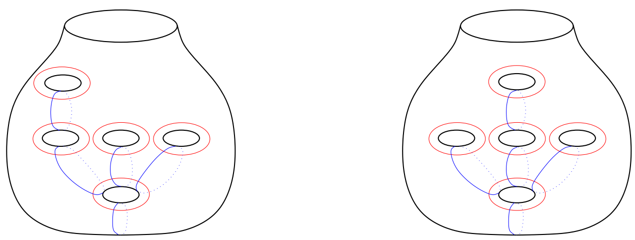

Figure 5: The surfaces [01112] and [01113] with the A-curves (red) and the B-curves (blue)

CHAPTER 2

Measured Foliations

Since most slalom monodromies are pseudo-Anosov, we can try to find the two invariant measured foliations. In this chapter we will show, that there is an explicit way to construct the measured foliations for positive tree-like mapping classes out of the rooted planar tree.

1. Tree-like train tracks

First we will introduce special train tracks, that will be needed in the following.

A *tree-like train track* is defined by a rooted planar tree. First construct the fiber surface corresponding to the tree, and then locally do the following replacements:

(1) The trunk vertex will be replaced by the following partial train track

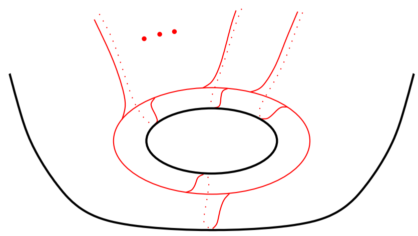

Figure 1: Replacement of the trunk vertex

(2) An interior vertex is replaced by the partial train track below

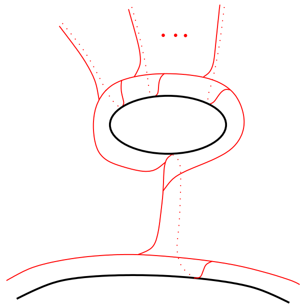

Figure 2: Replacement of an interior vertex

(3) The crown vertices are replaced the following way

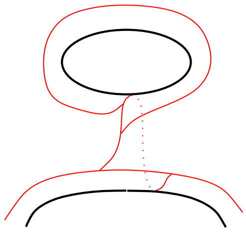

Figure 3: Replacement of an crown vertex

For our purpose a train track will be a branched submanifold on the surface with only two to one branchings up to isotopy together with a weighting of its arcs by positive real numbers. At the branchings the weights satisfy the first Kirchhoff rule. For a more general notion of train track see [**PH**].

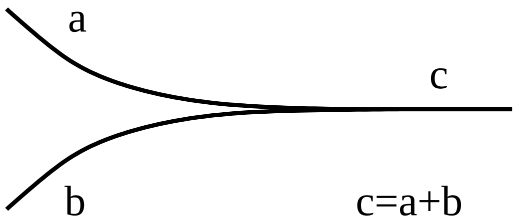

Figure 4: Kirchhoff rule at a branching point of valence three

For the tree [0111] this construction gives the following train track on the surface:

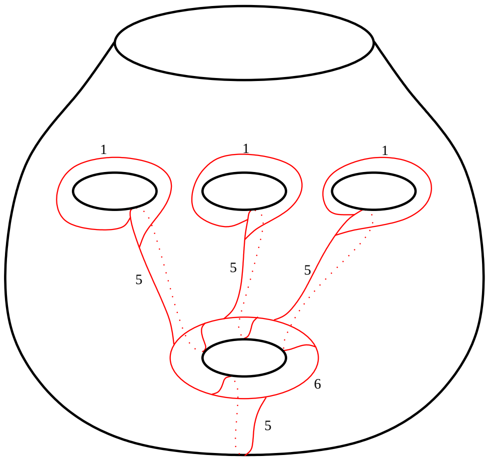

Figure 5: Tree-like train track with weights on the surface [0111]

In the above example we have chosen a minimal notation. By the Kirchhoff equalities the measures of all the other branches are defined uniquely. On a tree-like train track there exist arcs, that correspond to exactly one A- or one B-curves. The measures of arcs that correspond to the same A- or B-curve are equal. This can be verified by the Kirchhoff rule at the branching points. The arcs on the train track, that don't correspond to exclusively one A- or one B-curve are called *bridges*. The train track is homotopic to the union of the A- and B-curves, whereas the bridges are contracted to points by this homotopy.

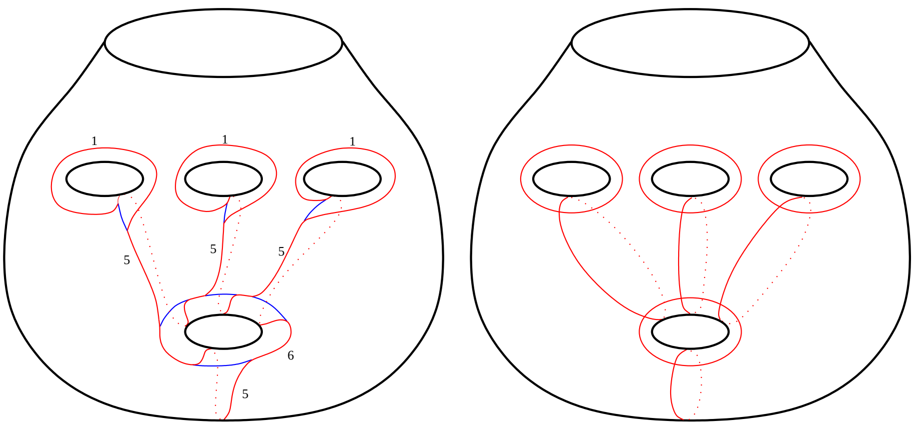

Figure 6: On the left side bridges on the train track [0111] marked blue, on the right side the same surface with the A- and B-curves

LEMMA 1.1. *A tree-like train track on a surface of genus g with one puncture can be labelled minimally by $2g$ numbers as shown in figure 8. Each number corresponds to an A- or a B-curve, in other words, each number corresponds to a vertex or an edge in the rooted planar tree.*

PROOF. Studying local pictures of the train track, it can be easily verified, that the labelling is minimal. Begin with the crown vertices and work downwards until reaching the trunk. □

LEMMA 1.2. *A tree-like train track on a surface of genus g provides a unique element up to sign in the homology of the surface.*

PROOF. Take as a basis for the homology the system of all A- and B-curves oriented in the following way: Chose an orientation for the A-curve corresponding to the trunk. Then orient any B-curve, such that it intersects any A-curve positively. In the previous Lemma we have seen, that to each A- and B-curve the train track assigns exactly one number. Taking these numbers as coefficients of the corresponding cycle, we get an element in the homology. □

2. Finding the measured foliation

We start with a disjoint union of simple closed curves γ, and let act the monodromy T iteratively on these curves. We will define $\gamma_n := T^n(\gamma)$ for $n\epsilon\mathbb{N}$. For every n, γ_n is again a disjoint union of simple closed curves and a measured foliation, where the transverse measures are given by the minimal intersection number with the above curves. When we let go n to infinity, the limit of the γ_n as a measured foliation will be the stable Thurston foliation.

We chose the union of all B-curves for γ and let act $T = T_A \circ T_B$ on them.

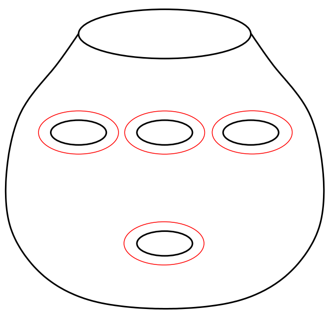

Figure 7: The fiber surface with the curve γ

After the second iteration we have the following picture:

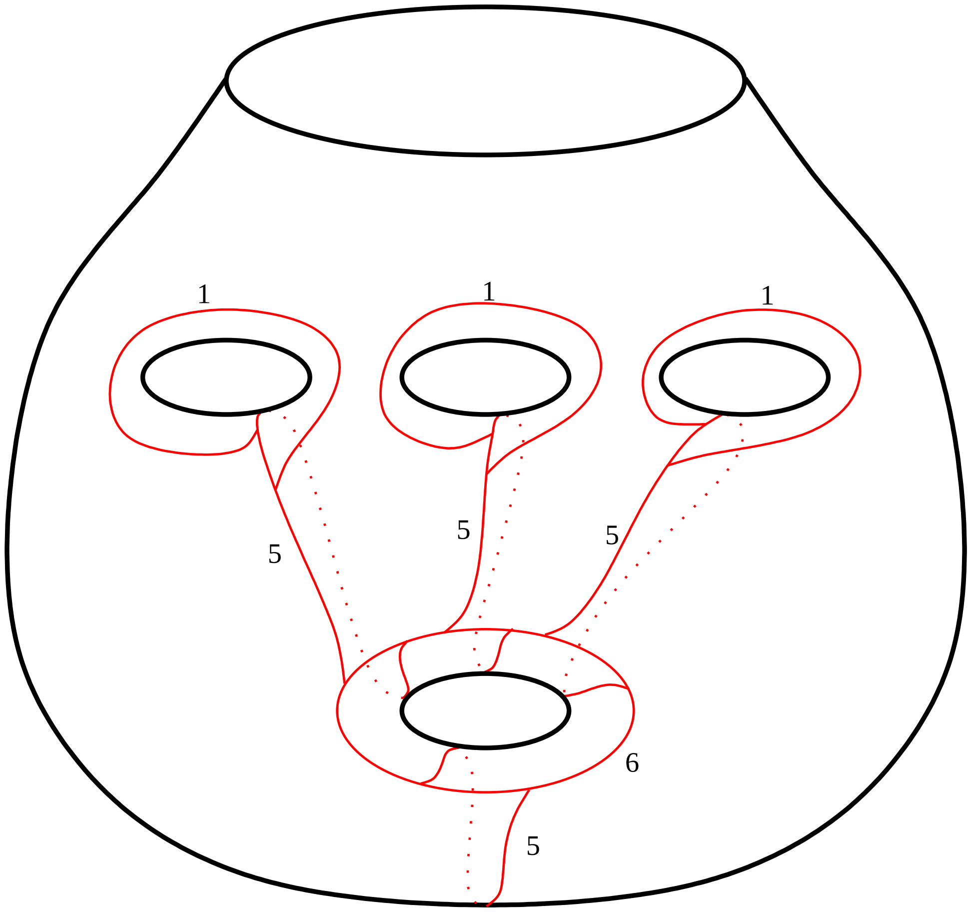

Figure 8: The fiber surface with the train track representing $T(\gamma)$

We have used a train track to represent the union of simple closed curves, the coefficients represent the number of arcs. We see, that the train track we get is a tree-like train track as defined in the previous section. When we let act T on this train track again, we see, that the train track is invariant under T. To be more precise, we get a new train track, and after isotopy and collapsing, we get again a tree-like train track but with new weights. We get new integer coefficients for the arcs of the train track. Having a closer look, we see that the new coefficients are built from the old ones by an integer matrix. We label the arcs that correspond to vertices by $\{x_i\}$ and those who correspond to the edges by $\{y_i\}$. In this example we have $1 \leq i \leq 4$. The trunk vertex is x_1 and the trunk is y_1. Then the indices increase from bottom to top an from left to right, as introduced in chapter 1. By symmetry we have $x_2 = x_3 = x_4$ and $y_2 = y_3 = y_4$ in the above example. The linear equations to get the new coefficients $\{x_i'\}$ and $\{y_i'\}$ of the train track after the action of T are:

$$\begin{aligned}
x_1' &= y_1 + y_2 + y_3 + y_4 - x_1 = y_0 + 3y_2 - x_1 \\
x_2' &= y_2 - x_2 \\
y_1' &= x_1' - y_1 \\
y_2' &= x_1' + x_2' - y_2
\end{aligned}$$

We don't get only positive signs in the equations, since every Dehn twist changes the train track, and we have to pull back some strands, in order to get a minimal train track.

So the train track represents the stable measured foliation. The linear equations above give us an integer matrix. The dominant eigenvalue and the associated eigenvector correspond to the geometrical growth and the measures of the arcs in the train track.

In the example above, we see the idea of the algorithm. As invariant train track, we obtain the tree-like train track associated to the rooted planar tree as defined in the above section.

The transverse measures on the arcs are given by solving an eigenvalue problem of an integer matrix. The matrix is given by the action of the monodromy on the measures of the edges and the vertices of the tree.

We give an explicit way to build the matrix M_T:

The matrix M_T is the product of the matrices M_A and M_B that describe the action of the diffeomorphisms T_A and T_B on the edges $\{y_i\}$ and the vertices $\{x_i\}$, $1 \leq i \leq g$. M_T, M_A and M_B are integer $2g \times 2g$ matrices, where g is the genus of the fiber surface. We denote the tree vector introduced in chapter 1 by β. We write the images of the vertices and edges in the rows of the matrices. The images of the edges and vertices are given by:

$$\begin{aligned}
M_A(x_i) &= x_i \\
M_A(y_1) &= x_1 - y_1 \\
M_A(y_i) &= x_i + x_{\beta[i]} - y_i, \quad i > 1 \\
M_B(y_i) &= y_i \\
M_B(x_i) &= y_i - x_i + \sum_{l,\beta[l]=i} y_l
\end{aligned}$$

The matrix M_T is given by:

$$M_T = M_A \circ M_B.$$

THEOREM 2.1. *The two invariant transverse measured foliations for a monodromy diffeomorphism of a slalom knot, whose Dynkin diagram is not among A_{2n}, E_6 or E_8, are constructed in the following way: The stable measured foliation is given by a the tree-like train track constructed out of the rooted planar and the transverse measures and the geometrical growth are given by the dominant eigenvalue and the corresponding eigenvector of M_T. The unstable foliation is given by applying C on the stable foliation, where C is an involution that conjugates T to its inverse.*

PROOF. To prove this theorem, we have to show, that the tree-like train track is invariant under the monodromy T, and that the coefficients behave the way we predicted with the Matrix M_T. Therefore we study local pictures of the train track and observe how the action of the monodromy changes this local pictures. We will always perform one positive Dehn twist and observe what happens to local pictures on the train track. We will show the new local train tracks after isotopy and after collapsing parallel arcs.
In a first step we will look at the action of the twists T_B on the train track. Since the these twists correspond the vertices of the tree, we only have to check neighborhoods of the vertices. Let's look at the trunk vertex:

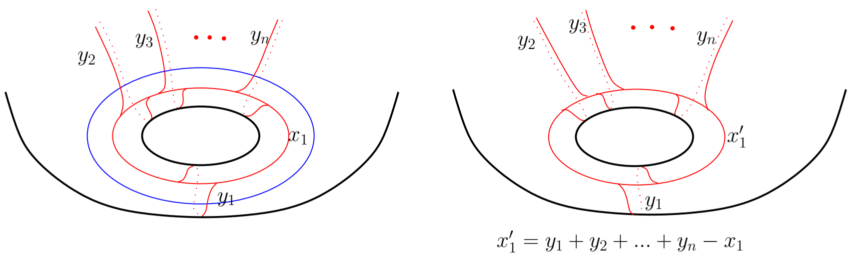

Figure 9: The trunk vertex under the action of a twist of type B (blue curve)

The neighborhood of an interior edge will change the following way:

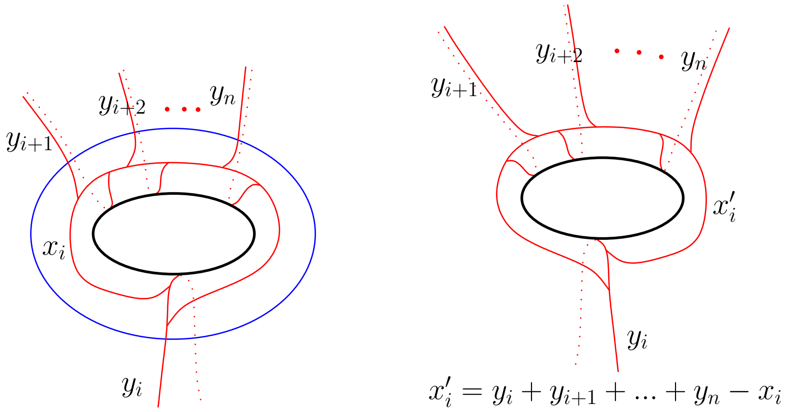

Figure 10: An interior vertex under the action of a twist of type B (blue curve)

And at last let's look at the crown vertex:

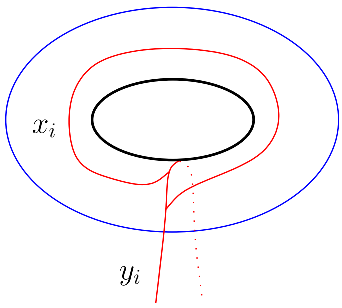

Figure 11: A crown vertex under the action of a twist of type B

Now we look at the action of the twists T_A. Analogous we only have to check the neighborhoods of the edges, since the A-curves correspond to the edges in the tree.
Let us look at the trunk:

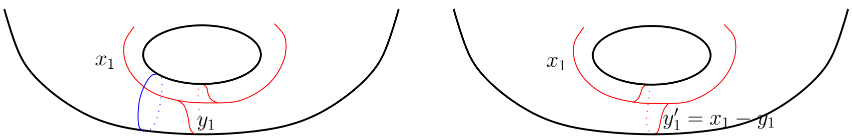

Figure 12: The trunk under the action of a twist of type A (blue curve)

The other edges change the following way:

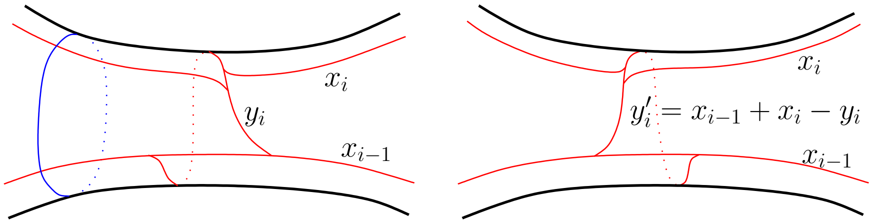

Figure 13: An arbitrary edge under the action of a twist of type A (blue curve)

Now we see, that the train track in neighborhoods of the vertices and the edges after performing $T = T_A \circ T_B$ look the same as before. So the tree-like train track won't change qualitatively under the diffeomorphism. This train track represents the stable foliation, since the width of the arcs in the train track is getting increased by the action of the monodromy.

By observing the action on the train track, we see, that the matrix we constructed above describes the action on the weight of the arcs, again we work in the neighborhoods of vertices and edges.

The matrix M_T, defined as above, characterizes the action of the monodromy on the weights of the train track. The train track is invariant and the ratio of the weights will converge to a set of weights, that belong to the invariant measured foliation. So this weights correspond to a positive eigenvector with a positive dominant eigenvalue. Having labelled the arcs of the train track by weights, that are coefficients of the eigenvector, the monodromy action is a multiplication of the weights by the eigenvalue. This eigenvalue is the geometrical growth of this monodromy diffeomorphism.

So the train track is uniquely determined up to a positive factor and thus we know the measured foliation. Now we have defined the stable foliation, what we need is the unstable foliation too. For this, we can do the same algorithm with the inverse of the monodromy T^{-1}, or we can use the involution C, with $T^{-1} = C \circ T \circ C$ (see next chapter) and C of the stable foliation is the unstable foliation. In the end we get the pair of invariant transverse measured foliations.

□

Since the homology is a quotient of the fundamental group it follows that $\lambda_{hom} \leq \lambda_{geom}$. A'Campo showed in [**AC1**] that $1 < \lambda_{hom}$ for all slalomknots, whose Dynkin diagram is not among A_{2n}, E_6 or E_8. As a corollary we get the following statement:

THEOREM 2.2. *For a slalom knot monodromy the geometrical growth equals the homological growth:*

$$\lambda_{geom} = \lambda_{hom}$$

PROOF. Lemma 1.2 tells us, how a tree-like train track is associated to an element in the homology. The invariant train track, as well as the corresponding element in the homology, are stretched by the same factor, the homological growth λ_{hom} and the geometrical growth λ_{geom}, respectively. λ_{geom} is the positive dominant eigenvalue of the monodromy matrix M_T. For this specific element in the homology, coming from a tree-like train track, the growth is λ_{geom}. Since we have the inequality $\lambda_{hom} \leq \lambda_{geom}$, and we have found an element in the homology, that is stretched by λ_{geom}, we get that $\lambda_{geom} = \lambda_{hom}$. □

For tree-like mapping classes, where positive and negative Dehn twists occur, the above statement is not true. A class of such examples can be found at Brinkmann (see [**Br**]). These diffeomorphisms arise from a tree with no branchings, corresponding to a surface with even genus. The corresponding tree vector equals $[012...(2n-1)]$ and the geometrical Dynkin diagram is A_{4n}. In contrast to our diffeomorphisms, the Dehn twists corresponding to the first n vertices and edges are positive ones, and the last $2n$ Dehn twist are negative ones. In this class of examples, the homological growth is not realized by a eigenvalue of the homological monodromy, and the homological and the geometrical growth are not equal. Furthermore, as n increases, the geometrical growth decreases and converges to one.

In fact, we have only drawn the invariant train track of the foliation. We still don't have the picture of the foliation and its singularities. To get this, we can cut the surface along all A-curves. We still have a connected surface. Then we can chose some more curves to cut along, so that we have decomposed our surface into a collection of pair of pants. Since we know the measures, we can draw on each pair of pants a measured foliation (on a pair of pants, there are only 6 qualitatively different possibilities to draw a measured foliation). And then we have to glue back the pair of pants, so that we get back our original surface, now with the measured foliation. We see, that along all curves, that we have cut the surface, that don't belong to the set of the A-curves, the singular leaves get identified. Since the surface minus A-curves is connected, we can do Whitehead-moves so that we get only one singularity, that is located on the boundary.

THEOREM 2.3. *For slalom knots, whose Dynkin diagram is not* A_{2n}, E_6 *or* E_8*, the foliation has a single* $4g$*-prong singularity that is located on the boundary, where two singular leaves belong to the boundary.*

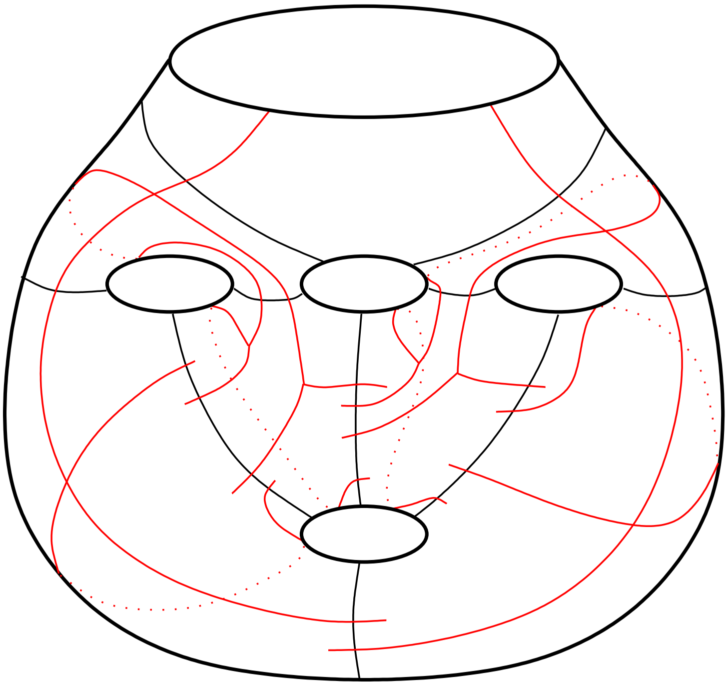

Figure 14: The singular leaves for the stable measured foliation in the example [0111]

For the examples of Brinkmann the above theorem does not hold. The invariant measured foliations have more than one singularity.

We can now improve our picture of the surface with the measured foliation on it. For this, we cut the surface along all B-curves. We get a sphere with $2g+1$ holes, where the central hole is the original boundary, where the singularity is located. We arrange our picture in a way, that the original boundary is in the middle an the other boundary components are arranged symmetrically, according to the tree:

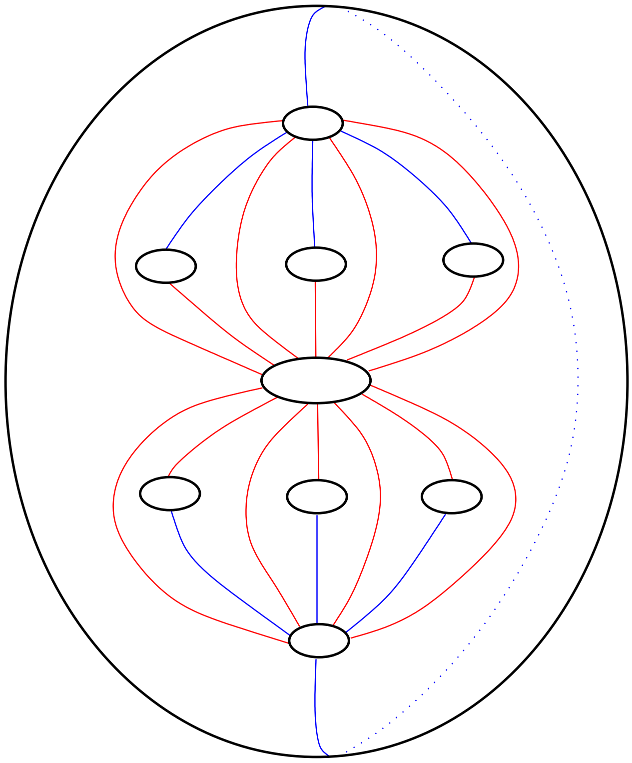

Figure 15: The picture of the surface cut along the B-curves with the singular leaves (red) and the A-curves (green)

The $4g-2$ singular leaves are placed the following way: The A-curves are now $2g-1$ arcs. Each singular leaf starts in the original boundary curve and goes to one of the new boundary curves.

To each new boundary component there goes a singular leaf from the boundary. We stop drawing the singular leaf, when it reaches the first B-curve. Of course, the singular leaves don't intersect pairwise. Each singular leaf goes from the boundary to a B-curve without crossing an A-curve and two singular leaves are not isotopic. These constraints give us an unique way of drawing the singular leaves in this picture.

To know the complete picture of the measured foliation, we need the information, how the B-curves have to be glued together. The measure of a B-curve is the sum of the measures of all adjacent edges to the corresponding vertex. We put the two B-curves together, such that the singular leaves of the two sides coincide and before gluing them, we twist counterclockwise sucht that the measure between the two corresponding singular leaves on this B-curve is the measure of the corresponding vertex.

CHAPTER 3

Involutions

1. T is strongly inversive

The monodromy of slalomknots has a very special property: the monodromy diffeomorphism is strongly inversive (see [**AC3**]). Strongly inversive means, that there exists an involution C such that $T^{-1} = C \circ T \circ C$. This property has been inherited by the knot, that is itself strongly invertible. A knot K in S^3 is called strongly invertible, if there is an involution of (S^3, K) which preserves the orientation of S^3 and reverses the orientation of K (see [**K**]). Slalom knots are fibred knots and Tollefson showed in [**To**] that the monodromy inherits this kind of property and becomes strongly inversive.
Obviously, there doesn't exist only one such involution. For every diffeomorphism D, that commutes with T, $TD = DT$, and which is strongly inversive by C, DC is also an involution with the above property:

$$\begin{aligned} DCDC &= D(CDC) = DD^{-1} = Id \\ DCTDC &= (DC)^{-1}TDC = CD^{-1}TDC \\ &= CD^{-1}DTC = CTC = T^{-1} \end{aligned}$$

Since T is pseudo-Anosov, the only diffeomorphisms that commute with T are powers of T and elements of finite order (see [**M**]), so $D = T^n$, for $n \in \mathbb{Z}$ or $D^m = Id$ for $m \in \mathbb{N}$. But since our special pseudo-Anosov diffeomorphisms have only one singularity located on the boundary, a finite order element that commutes with T and fixes the boundary can be only the identity. So the only diffeomorphisms that commute with our T are powers of T.
Up to conjugation of the pair (T, C) there are at most two involutions, since all $T^{2n}C$ and $T^{2n+1}C$ are conjugate among each other:

$$\begin{aligned} T^{2n}C &= T^nT^nC = T^nCCT^nC = T^nCT^{-n} \\ T^{2n+1}C &= T^nTT^nC = T^nTCCT^nC = T^n(TC)T^{-n} \end{aligned}$$

The next question to answer is, if the pairs (T, C) and (T, TC) are conjugate.

2. Are C and TC conjugate?

To answer this question we will need a special property of C and we will check, if TC has the same property.

2.1. The fixcurve for C. We fix the involution C to be the one, that comes from the involution in the unit disk given by $(x, v) \mapsto (x, -v)$. On the surface C is a reflection S (front and back are changed on the surface) followed by a left Dehn twist along all B-curves:

$$C = T_B^{-1} \circ S$$

That C is an involution for the monodromy $T = T_A \circ T_B$ with the above properties, is shown by the little calculation below:

$$CC = T_B^{-1} S T_B^{-1} S = T_B^{-1} (S T_B^{-1} S) = T_B^{-1} T_B = Id$$
$$CTC = T_B^{-1} S T_A T_B T_B^{-1} S = T_B^{-1} S T_A S = T_B^{-1} T_A^{-1} = T^{-1}$$

For the involution C there exists a curve γ_0 that is fixed pointwise (see [**AC4**]). In the unit disk, it is the curve that consists of all points (x, v), with $v = 0$. On the surface, it is a curve that goes from boundary to boundary along the symmetry axis of the reflection S. Since γ_0 is fixed pointwise by S and does not intersect any B-curve, it is fixed pointwise by C.

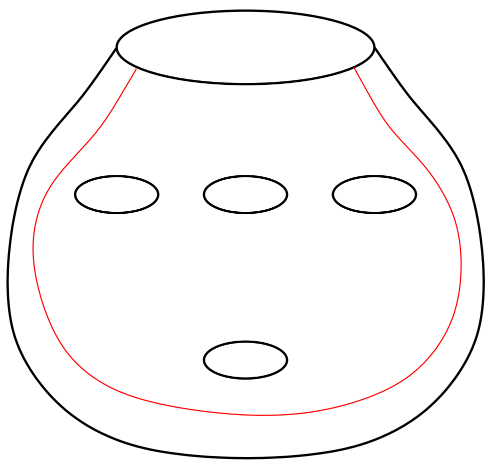

Figure 1: The fixcurve γ_0 on the surface [0111]

2.2. The Fixcurve for TC. Above we have fixed the monodromy T and the involution C on the surface. So we get for the involution TC

$$TC = T_A T_B T_B^{-1} S = T_A S.$$

In [**AC4**] it is shown, that for TC there also exists an arc γ_0', that is fixed pointwise. γ_0' is constructed out of γ_0 and the A- and B- curves in the following way:

- Go along γ_0 until you cross the first A-curve (this will of course be the trunk).
- Go along this A-curve in a right Dehn twist way until you cross a B-curve.
- Go along this B-curve in a left Dehn twist way until you cross the next A-curve.
- Repeat these two steps until you cross the trunk curve again.
- Then you follow again γ_0 (the part you didn't already follow) to the boundary.

It is easy to see, that this curve is fixed by TC, looking at local pictures around an edge.

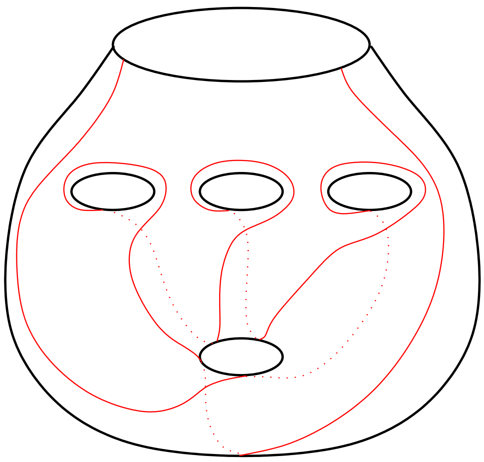

Figure 2: The fixcurve γ_0' on the surface [0111]

2.3. A method to distinguish C from TC. First we change the fixcurves, so that they become simple closed curves embedded in the interior of the surface. So we push both curves out of the boundary and call them γ and γ' respectively.

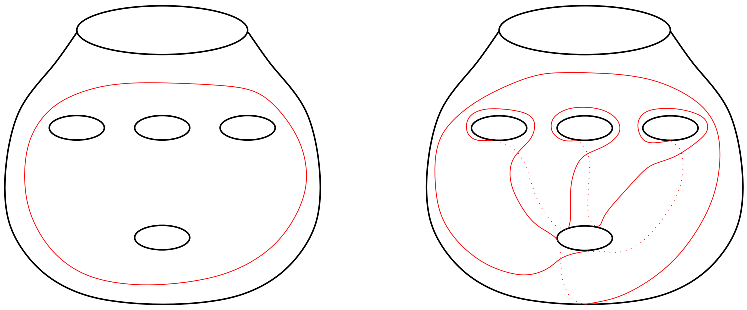

Figure 3: The curves γ (left) and γ' (right) for the surface [0111]

Now we get two triples (T, C, γ) and (T, TC, γ'). The important thing we observe, is that γ and $T(\gamma)$ intersect exactly in one point. The idea to separate C from TC is to show, that the curves γ' and $T(\gamma')$ intersect in more than one point.

LEMMA 2.1. *The curve γ constructed above and its picture under the monodromy T intersect in exactly one point.*

PROOF. We can argue in the standard picture as above, since intersections are preserved under conjugation. We see, that in any case the curve γ intersects with only one curve of the tree, and that is the A-curve which comes from the trunk a_1. Thus $T(\gamma) = T_A(T_B(\gamma)) = T_A(\gamma) = T_{a_1}(\gamma)$. Therefore the above statement is proved. □

We know now, how γ and $T(\gamma)$ intersect, namely in exactly one point. In the following, we will see, that γ' and $T(\gamma')$ intersect in most cases in more than one point, and so the two pairs (T, C) and (T, TC) cannot be conjugate.

We will prove the non-conjugacy of the two involution for a smaller class of trees, those that have at least 3 crown vertices. For the further observation those trees that have only one or two crown vertices are not interesting for us, since one crown vertex trees produce non pseudo-Anosov monodromies and two crown vertex trees have only one planar embedding up to congruence.

We will subdivide our class of trees with at least 3 crown vertices in two groups. Group one contains all trees, that have at least one interior vertex of valence more or equal than four. Group two contains all trees, that have only interior vertices of maximal valence 3.

First let us look at group one. The minimal representative for this group is the tree [0111].

LEMMA 2.2. *For all trees, that have at least one vertex of valence four, the curves γ' and $T(\gamma')$ intersect in at least three points.*

PROOF. First, we analyse our minimal representative [0111]:

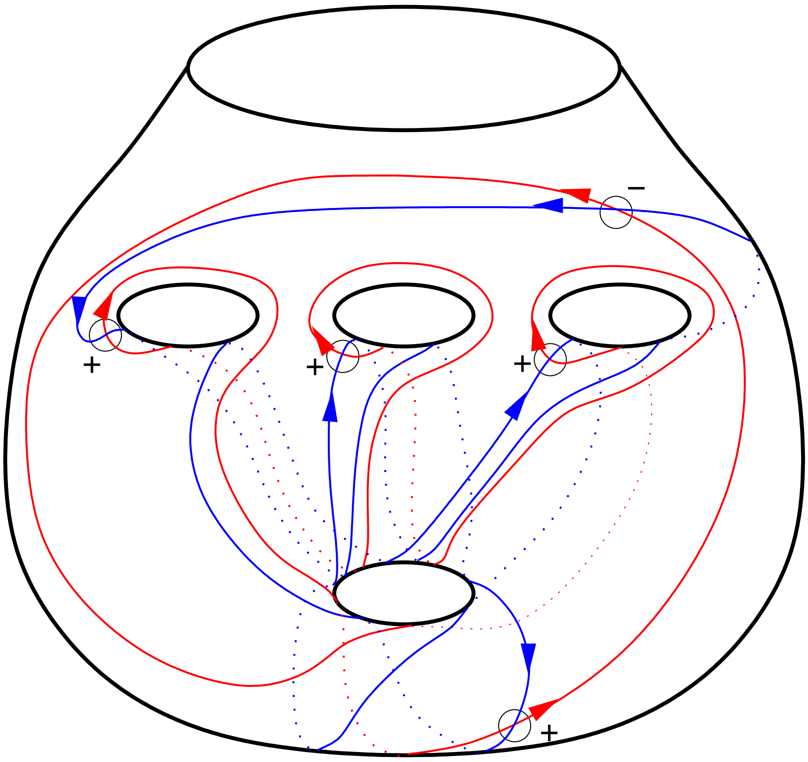

Figure 4: The signed intersection points of γ' (green) and $T(\gamma')$(blue)

In the example, it can be seen, that the homological intersection number equals 4-1=3. So we can be sure, that these three intersection points can not be cancelled by isotopy of the two curves. If [0111] can be found as a subtree, then these three intersection points always remain, some more intersection points can occur. So we can be sure, that if in a tree we have at least one vertex of valence four, that is equivalent, that this tree has [0111] as a subtree, then γ' and $T(\gamma')$ intersect at least in three points. □

Let us look now at the second group of trees, those who have at most three valent vertices, but at least three crown vertices. The minimal representative for this group is the tree [01122].

Lemma 2.3. *For all trees, that have at least three crown vertices but no vertices of more than valence three, the curves γ' and $T(\gamma')$ intersect in at least three points.*

Proof. Again we have to analyse our minimal representative carefully, to understand what happens in the general case:

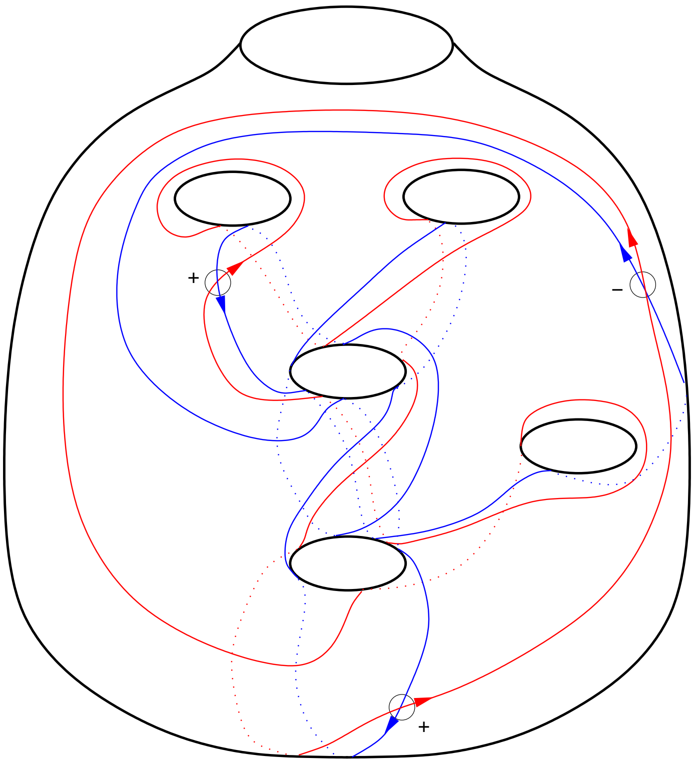

Figure 5: The signed intersection points of γ' (green) and $T(\gamma')$(blue)

For the tree [01122] we get three intersection points, but if we count them with their signs we only get 2-1=1. So we have to take a closer look at this situation and check, if two intersection points with opposite sign do cancel. For this we have to solve a pair of intersection points. We get two simple closed curves, and we have to show, that none of these two curves is non essential. In fact, every curve obtained by solving two intersection points is a non-separating curve for the surface, and therefore, two intersection points cannot cancel.

Having this, we see, that by enlarging our representative [01122] by subdividing edges or by attaching new edges, these three intersection points remain. Moreover, sometimes more intersection points will appear. The argument remains the same for this whole class of trees in group two. Hence the statement of the lemma has been verified. $\square$

Hence we have shown, that, if we start with a tree with at least three crown vertices, the fixcurve γ' of TC produces at least three intersection points with the curve $T(\gamma')$. On the other hand we have shown, that the fixcurve γ of C intersects always once with its picture $T(\gamma)$ under the monodromy. Therefore the pairs (T, C) and (T, TC) cannot be conjugate. We have found even more, we have found a criterion to distinguish (T, C) from (T, TC).

THEOREM 2.4. *For trees with at least three crown vertices the pairs* (T, C) *and* (T, TC) *are not conjugate.*

With this we get the following statement:

THEOREM 2.5. *Slalom knots, that arise from rooted planar trees with at least three crown vertices, are strongly invertible knots with exactly two involutions.*

CHAPTER 4

Reconstructing the planar tree

In this chapter we will establish an algorithm to reconstruct the planar tree out of the monodromy. We will need the triple (T, C, γ) from the last chapter. In the whole chapter we will only consider monodromies that arise from trees with at least three crown vertices.
Additonaly we will need also an operation to solve crossings of oriented curves. We will call the operation shown in figure 1 an anti-oriented solving of a crossing. The result gives an non-oriented curve. The curve will be oriented afterwards, following several rules.

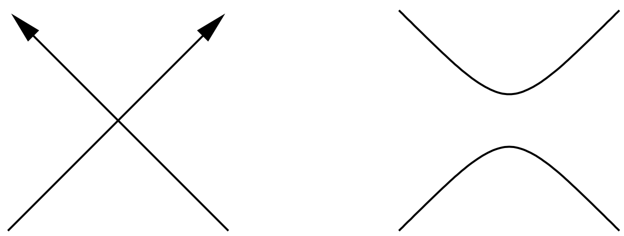

Figure 1: anti-oriented solving of a crossing

1. The algorithm

We start with the triple (T, C, γ). We will illustrate the algorithm in the standard situation. We can do this, since the operations we perform are invariant under conjugacy.

- We chose an arbitrary orientation on γ.

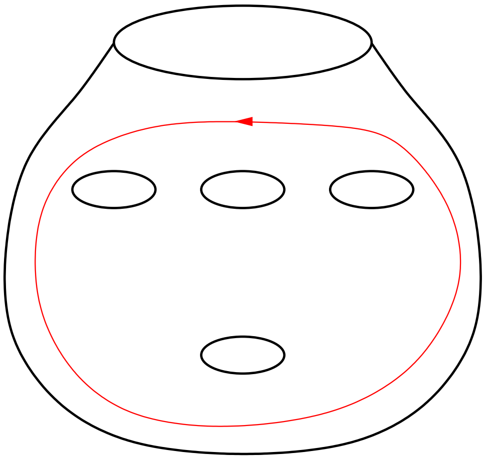

Figure 2: The oriented curve γ on the surface [0111]

- We let act the monodromy T on the oriented curve γ and get a new simple closed and oriented curve γ'. It is $T(\gamma) = T_A(T_B(\gamma)) = T_A(\gamma) = T_{A_0}(\gamma) = \gamma'$, since γ intersects no B-curve and only crosses the trunk-curve A_0.

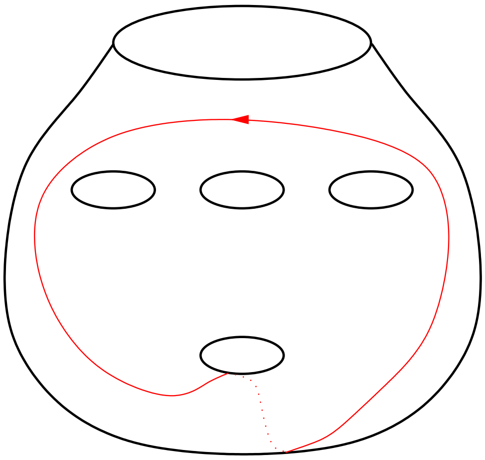

Figure 3: The oriented curve γ' on the surface [0111]

- The two oriented curves γ and γ' intersect in exactly one point. We eliminate this crossing with the above defined solving and get a simply closed curve a_1. We chose an orientation on a_1 such that the intersecion of γ with a_1 is positive. We have relocated the trunk curve.

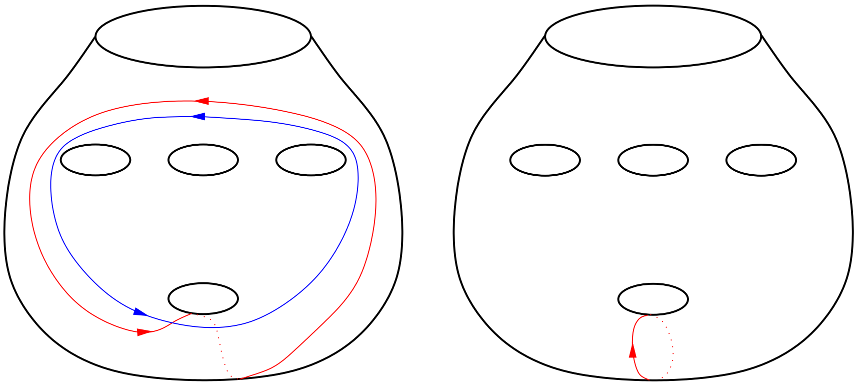

Figure 4: The surface with γ and γ' and the constructed twistcurve a_1

- We will continue our algorithm with the curve a_1. We let act T^{-1} on a_1.

$$T^{-1}(a_1) = T_B^{-1}(T_A^{-1}(a_1)) = T_B^{-1}(a_1) = a_1'$$

  The second equality holds, since a_1 intersects only B-curves. So we get a new simple closed, oriented curve a_1'.

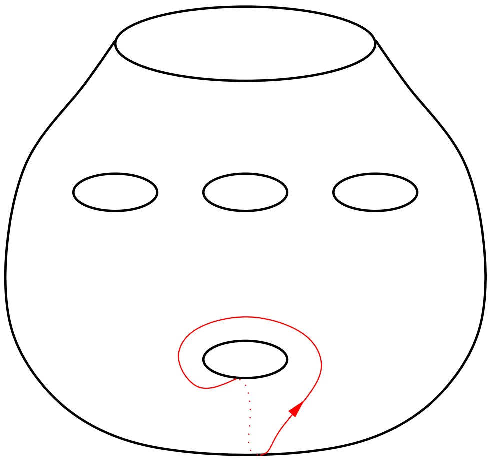

Figure 5: The curve a_1'

- Again a_1 and a_1' intersect only in one point, and this crossing we eliminate as before. We get an simple closed curve b_1. We chose an orientation on b_1 such that the intersection number of a_1 and b_1 is $+1$. The curve b_1 correspond to the trunk edge in the planar tree.

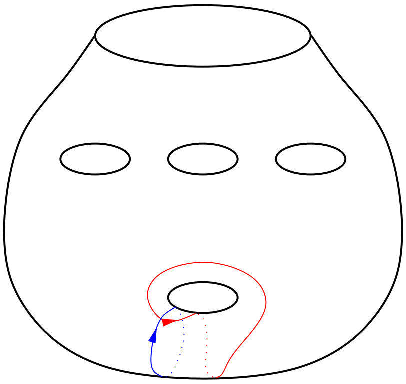

Figure 6: The surface with a_1 and a_1' and the constructed twistcurve b_1

- Next we let act T on the oriented curve b_1. We get a new curve b_1'.
$$T(b_1) = T_A(T_B(b_1)) = T_A(b_1) = b_1'$$
The new curve b_1' intersects b_1 in n_1 points, where n_1 is the number of adjacent edges of the vertex that corresponds to b_1 in the tree.

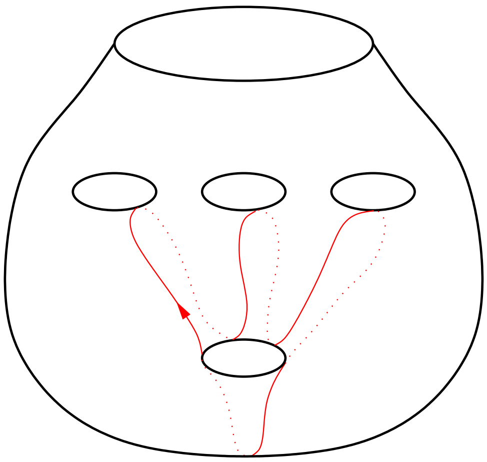

Figure 7: The curve b_1'

- Now we solve each crossing of b_1 and b_1' with an anti-oriented solving. So we get n_1 curves, which correspond to the adjacent edges to the vertex corresponding to b_1. We get the curve a_1, which we already know, and the we get also new curves a_2 to

a_{n_1}. b_1 intersects each curve a_i in exactly one point. We chose the orientation of the a_i such that the intersection number with b_1 is $+1$.

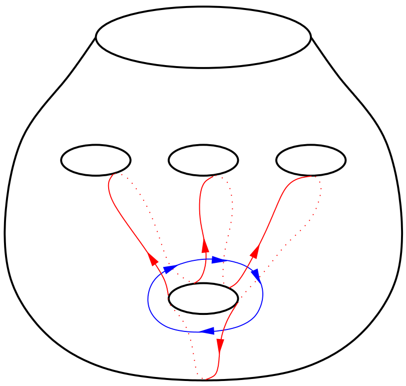

Figure 8: The surface with a_1 to a_{n_1}

- We continue with the curves a_2 to a_{n_1} by doing the same as before with the curve a_1: We let act T^{-1} on them. Then we get the curves b_1 to b_{n_1}. Each a_i gives two curves, b_1 which we already know, and a new curve b_i. The curve a_i correspond to an edge in the tree and b_1 and b_i to the two adjacent vertices.

Figure 9: The surface with a_i (left) and after the action of T^{-1} with a_i' (right)

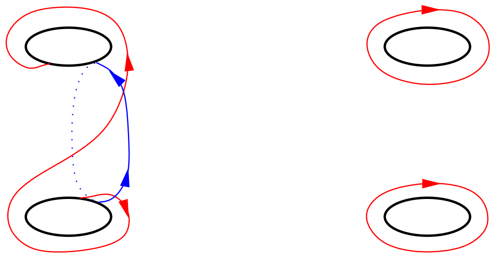

Figure 10: The surface with a_i and a'_i (left) and the surface with the curves b_1 and b_i after the anti-oriented solving (right)

- Now we can go on with $n_1 - 1$ new curves b_2 to b_{n_1}. With each of them we do the same as we did above with the curve b_1: We let act T on them. By doing this and after a crossing solving we get the curves a_2 to an_2, since we get all curves, that correspond to the adjacent edges to the edges that correspond to the vertices b_2 to b_{n_1}.

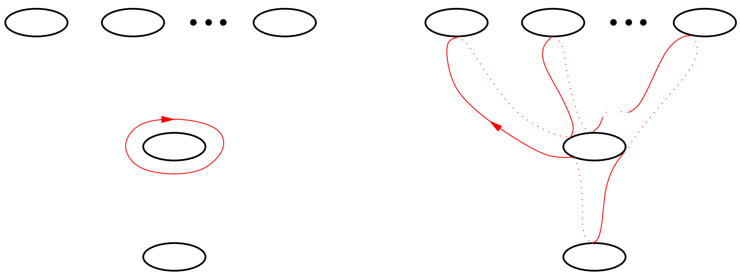

Figure 11: The surface with b_i (left) and after the action of T with b'_i (right)

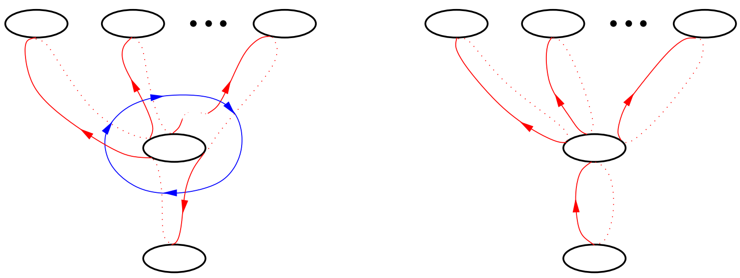

Figure 12: The surface with b_i and b'_i (left) and the surface with the curves a_i to a_j after the anti-oriented solving (right)

- We repeat the last two steps as long as we can create new curves, i.e. as long as a curve b_j gives more than the curve a_j. If b_j corresponds to a crown vertex, the algorithms stops in this branch. When we have arrived in all branches in a crown vertex, the algorithm stops. Then we have found back all A- and B-curves for the monodromy and we can reconstruct the *planar* tree, since the cyclic ordering of the A-curves around a B-curve gives us the information for the embedding of the tree into the plane.

The algorithm above gives us a tool to find back the planar tree out of the monodromy. So the planar tree is an invariant for the knot. The consequence is, that two slalom knots, which come from the same abstract rooted tree with different embeddings into the plane, can be distinguished. This gives us the following statements:

THEOREM 1.1. *For a positive tree-like mapping class, coming from a rooted planar tree with at least three crown vertices, the rooted planar tree can be reconstructed with the algorithm established above.*

COROLLARY 1.2. *Two positive tree-like mapping classes coming from different planar trees are not conjugate.*

PROOF. For mapping classes coming from rooted planar trees with at least three crown vertices, the corollary is a direct consequence of the theorem. For mapping classes with two or one crown vertex, there exists only one planar embedding up to congruence. The knots corresponding to this mapping classes have already been classified by the theory of Montesinos links (see [**Tu**]). ☐

As a direct consequence we get the next statement:

COROLLARY 1.3. *Two slalomknots that come from non-congruent rooted planar trees are different.*

Thus the construction of knots via rooted tree generates a lot of different but mutant knots, since non-congruent embeddings of the same abstract tree correspond to mutations of the knot.

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